What We Call Middle School MATHEMATICS

FRANCIS GARDELLA

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DEDICATION

To My Family

Twice in my life, I have received a standing ovation from a group. Each time, when quiet ensued, the first thing I did was to ask the group to "do me a favor,' stand and give my wife, Gail, a standing ovation. For it is Gail who truly deserved the ovations and many more. She has been the one who has made whatever I have all possible. Her way of making things better no matter the circumstances has truly been an inspiration for me lo' these many years.

Always lighting the way

Always the love of my life

To my children, Jennifer and Derek and their spouses, Jim and Laura, the constant reminders of how fortunate I am to be a parent of such wonderful adults. And to my granddaughters, Allie, Vicki, Stephie, Paige and Claire, (listed in the order in which I have met them as they entered the world) of whom I am so proud for their accomplishments and grateful that they still accept and smile at my 'grandfatherly ways.'

To a group of men who included me in what was dubbed at Fordham the 'family.' For whenever you saw one of us, there were usually at least two others around somewhere. So to Art, Bill, Charlie, Jack, Tim and Tom as well as Naps who left us some time ago. The idea that we met more than 60 years ago as strangers and continue today as friends is and always will be a very special part of my life.

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ACKNOWLEDGEMENTS

The ideas in this book, my other books and, in fact my whole career, would not exist without those who have given me the time out of their lives and have had an incredible impact on me as a person, as a teacher and as a writer. And while I list them here in different 'categories' of my life and career, for me they all meld into what I call my life.

These acknowledgements must begin with J. R. Williams, someone I met in my third year of teaching and became a life-long friend until he left us in 2006. Much of my knowledge about life, students and life in the classroom began with Jim those many years ago.

Theodore Brown, Bob Mulligan and Fran DeZort are three principals for whom and with whom I worked during my years as a classroom teacher. When I think of these men, I think of the idea 'No way could three people have been more different' and yet their impact showed me that very different roads can lead to being successful, a life lesson I have never forgotten. With these, Dorothy Roberts, my mathematics mentor/supervisor for my classroom years was the quiet, calm and brilliant person who, by accepting my 'novel lessons', encouraged me to look at mathematics differently for the betterment of the children. And a summary of my classroom years would not be complete without the mention of Ted Stanik, a good friend and one of the most, if not the most creative mathematics teachers I have ever met.

Ernest Duncan of Rutgers University created the most profound change in my career. For with his guidance through my doctoral program, I was able to move in so many more different directions. "St. Ernest" as my wife still refers to him, was the cornerstone of my future at the university and in publishing. His mode was calm and his superb work ethic was unmatched. And he taught these to me without my knowing it at the time.

On the mathematics side, Jerry Porter King of Lehigh and Samuel Borofsky of Brooklyn College made me see that mathematics was doable, explainable and enjoyable if the instruction was good. And theirs was excellent.

My years in East Brunswick blessed me with two colleagues, Lorraine DeLuco and Carol Kolsky who took the many ideas that we generated and made them a reality for the children and the teachers.

In 'expanding' my work, I have to acknowledge the people at the Science Center at Fairleigh Dickinson University, Don Peck, Hank Gary and the inimitable Mal Sturchio who brought me in as the 'math guy' and let me feel that I was always part of the team.

And now I find myself still surrounded by brilliant educators. In my time at Hunter College, Sandi Clarkson, Bob Gyles, and Barbara Barone stand out as being my inspiration and support, two attributes that make me grateful that they entered my life. Together with them and my dear friend of so many years, Maria DeLucia, it is my hope that our work together continues for years to come.

And to my many students, sources of joy, pride and, at times, frustration (as any teacher will tell you.) Through these many years, they not only made me feel that my efforts worthwhile but also gave me the incentive to continue in my attempts to make both their lives and their life with mathematics better.

As you may realize, this series of books, which are in a sense my thoughts on mathematics and teaching, would not have been possible without my publisher, George Johnson who has encouraged me to write the books I have wanted to write and then made me see them in actuality.

To all of these fine people, it is my hope that I have been deserving of your time and care.

ABOUT THIS BOOK

I approach the mathematics in this book not as 'knower' of mathematics relating what I know but as an investigator/experimenter, looking at ideas, developing concepts and pointing to a symbolic system by which the ideas can be expressed.

The foremost drive of the book is that mathematics is in reality and it can be discussed using realistic physical models and terminology. From this basis, the symbolism and terminology by which mathematics is communicated throughout the world is developed.

As with the development of any language, the mathematical symbolism and terminology are developed as needed so that ideas and concepts brought forth can be discussed without confusion.

It is hoped that this inductive view allows teachers to develop lessons and explanations by which all children can understand the mathematics they are addressing and realize the development of the symbolism and terminology that can be used.

As with my other books that IAP has published, this is not a text book but more of a 'teachers' book. It is simply the way I look at mathematics when I am initially trying to pursue an idea myself or when I am initially addressing it with students. It is certainly not a formal view of mathematics. There are many who have and will continue to write much better material in this format. And it is not a writing about the many aspects of how one teaches mathematics.

What We Call Middle School Mathematics, pages xi–xii.

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This book is a way for teachers to take their formal view of mathematics and create, in a sense, a kinesthetic journey aiding them in communicating the mathematics to their students.

My hope is that the pages of this book, as with my other texts from Information Age Publishing, *Short Geometry Labs and Algebra for the Middle Grades*, can form a communication between students and their teacher at the very beginning of their study and from this, through the teachers' formal knowledge of mathematics, develop the knowledge and love of mathematics needed for success.

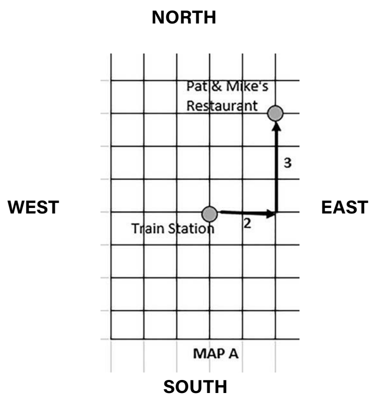
CHAPTER 1

HOW TO MOVE ON A GRID

1.1. MOVING AROUND THE CITY

It would be easy to find your way around a city if the streets formed squares or even rectangles. Look at Map A below. There is a train station where the trains bring you into the city. Directions to any place in the city are given by stating the number of blocks you walk.

For example, on the map, we have Pat and Mike's Italian Restaurant.



What We Call Middle School Mathematics, pages 1–7.

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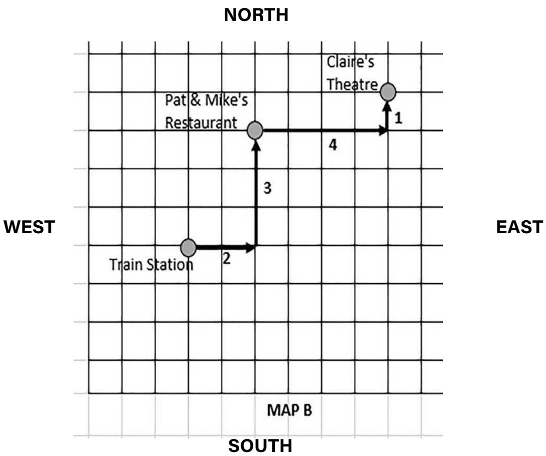
To move from the train station to Pat and Mike's Italian Restaurant, you would go 2 blocks east (to the right on the map) and 3 blocks north (toward the top of the map.) This is shown on Map A on page 1.



Map B shows Claire's Theatre. To move from the Pat & Mike's Italian Restaurant to the theatre, you would go 4 blocks east and then 1 block north. We can state this movement on the map from the restaurant to the theatre by using the pair $\overrightarrow{(4,1)}$ where the first number, 4, shows the movement east and the second number, 1, shows the movement north. The arrow, →, over the pair means it is a movement.

We call this type of pair of numbers an ordered pair because the order in which the numbers are written is important. For this map, the first number shows us the movement east. The second number shows us the movement north. If going from the restaurant to the theatre, you walked 1 block east and then 4 blocks north (written as $\overrightarrow{(1,4)}$ you would not be at the theatre.

Fill in the ____ that follow. The pair $\overrightarrow{(3,1)}$ would mean a movement of ____ to the east and a movement of ____ north.

Note: For those of you wondering, we will study the additional movements of west and south in a later chapter.

Summary:

- To move from the Train Station to the Restaurant you move $\overrightarrow{(2,3)}$.
- To move from the Restaurant to the Theatre, you move $\overrightarrow{(4,1)}$.

1.2. MOVING ON A MATHEMATICAL GRID

When we draw a 'map' in mathematics it is sometimes called a grid. On a grid, from a starting point P, you can move as many units to the right on the page and as many units up the page. (It all depends on how big your grid is.) As was stated earlier, we will discuss moving left or down the grid in a later chapter.

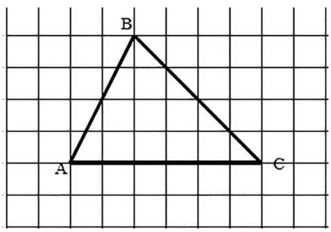
One of the first people to look at mathematics on a grid was Rene DeCartes who lived in the early 1600s. It was his idea (and all in mathematics follow this today) that if we have a pair of numbers, say $\overrightarrow{(3,5)}$, then the first number will tell us how many units we move to the right and the second number will tell us how many units we move up the page. Like with our map, this pair of movements is called an **ordered pair** because the first number always means a certain direction and the second number means a different direction.

Note: We call $\overrightarrow{(3,5)}$ a mover.

The movement $\overrightarrow{(3,5)}$ can be named with a lower-case letter. Let's use the letter '$\boldsymbol{v}\overrightarrow{(3,5)}$' to name the movement that uses the ordered pair. We can write $\boldsymbol{v}\overrightarrow{(3,5)}$ So $\boldsymbol{v}\overrightarrow{(3,5)}$ will move a point on the grid 3 units to the right and 5 units up the grid.

1.3. MOVING TRIANGLES ON A GRID

On the grid below, we have triangle ABC.



Grid 1-1

What we are going to do is move each of the points, A, B and C using the mover, $w\overrightarrow{(4,5)}$

When we move point A to the new point, we name the new point, P.

We can write, **"*w* moves point A to point P."**

When we move point B we call the new point Q.

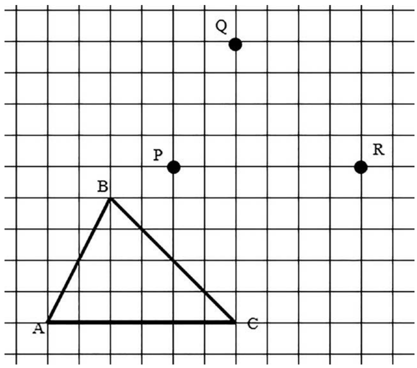
We can write, **"*w* moves point B to point Q."**

And when we move point C, we call the new point R.

We can write **"*w* moves point C to point R."**

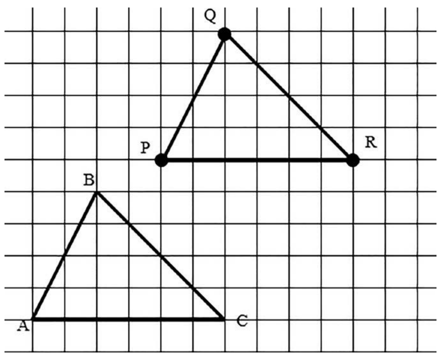
Since ***w*** moves something on the grid, we call ***w*** a **mover.**

Use Grid 1-2 to verify that the mover ***w*** moves each point, A, B,C 4 units to the right and 5 units up the page.



Grid 1-2

We now connect the three new points to form triangle PQR.



Grid 1-3

Look carefully at both triangles. What relationship do they have?

Measure the sides of each triangle.

What do you find?

Since we moved point B to point Q and point C to point R, the side BC is equal in length to side QR.

So if one side of triangle ABC has a certain measure, then there is a part of triangle PQR that will have that same measure.

So when we move a triangle with a mover, we create a new triangle that is the same as the original except it is in a different place on the grid.

The words we use: Although we have used the word ‘**mover’** in our work, **Mathematicians** agreed to use the word ‘**vector’** or **‘translation’** to describe the mover.

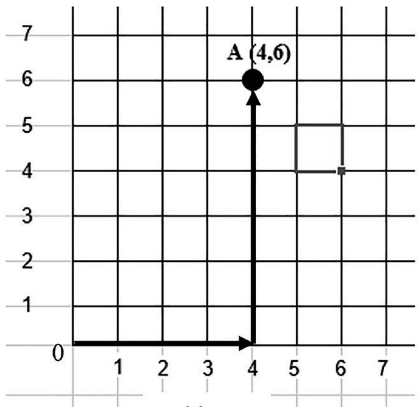
1.4. LOCATING POINTS ON A GRID

Ordered pairs can also be used to find an actual place on the grid. For example, from our map of the city, where can we say that Pat and Mike's Restaurant is?

To do this, we set up numbers on the grid as seen in Grid 1-4. We have a set of numbers along the bottom to count the spaces in a horizontal (↔) way and a set of numbers up the left side of the grid to count the spaces in a vertical (↕) way up the grid.

In locating points, we always begin at 0. So, beginning at 0, let's move 4 spaces horizontally (↔) and then 6 spaces vertically (↕).

Let's name this point A. To move from 0 to A, we moved 4 spaces horizontally (or to the right) and 6 spaces vertically (or up the grid.)



Grid 1-4

We show point A with an ordered pair A(4,6). The first number in the ordered pair shows that to go from 0 to point A, we make a horizontal move of 4 units and the second number shows that we make a vertical movement of 6.

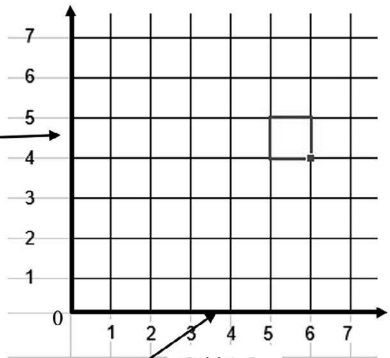
The difference between our work here and our work with vectors is that the movement from 0 gives this point a particular 'numerical name', A(4,6). Since this is a place and not a movement, we do not use an '→' with the ordered pair. (4,6) shows a single point on the grid and this is the only point on the gride with the ordered pair (4,6).

This is what Descartes did when he revolutionized the study of mathematics using grids.

1.5. A LINE ON A GRID

We can use the grid to see a 'picture' of an equation. To do this, we need to name two parts of the graph.

On Grid 1-5, we call the first vertical line the "**Vertical Axis.**" We will draw it darker than the other vertical lines.



Grid 1-5

We also name the first horizontal line the **"Horizontal Axis."** We will draw it darker than the other horizontal lines.

We use arrows, → at the end of these '**Axes**' (the plural of the word 'Axis') to show that both of the lines continue beyond 7.

So, let's show how we can use ordered pairs to locate points on a grid.

To find the numbers in the ordered pair, we will use the rule:

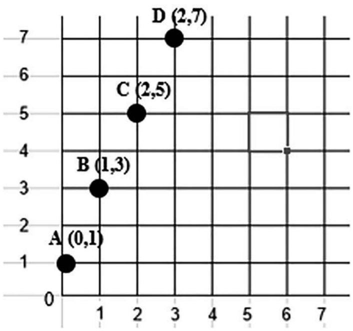
a. Choose a whole number from 0 to 3.
b. Multiply your number by 2
c. Add 1 to your answer in 'b.'

Let's use the letter 'N' to represent the number that we chose in 'a.'
Let's use the letter 'T' to represent the answer after we do part 'c.'
We create a table of values for this.
In this table of values, we have:

Column 1	Column 2	Column 3	Column 4	Column 5
Value of N	**Equation**	**Value of T**	**Ordered pair (N,T)**	**Point Name**
0	T = 2(0) + 1	1	(0,1)	A
1	T = 2(1) + 1	3	(1,3)	B
2	T = 2(2) + 1	5	(2,5)	C
3	T = 2(3) + 1	7	(3,7)	D

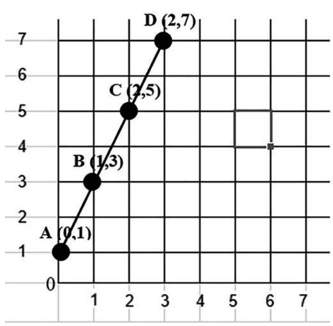
From the equation column, we can see that the equation used is T = 2N + 1.

Here is what we have when we put these ordered pairs on the grid.



Grid 1-6

And when we connect the points on the grid we get a line that goes from A to D.



Grid 1-7

So, this gives us a small picture for the equation, T = 2N + 1.

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