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ADVANCES IN ECONOMETRICS VOLUME 45A

ESSAYS IN HONOR OF JOON Y. PARK: ECONOMETRIC THEORY

EDITED BY

YOOSOON CHANG
Indiana University, USA

SOKBAE LEE
Columbia University, USA

And

J. ISAAC MILLER
University of Missouri, USA

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LIST OF CONTRIBUTORS

Chafik Bouhaddioui	United Arab Emirates University, UAE
Jean-Marie Dufour	McGill University, Canada
Jiti Gao	Monash University, Australia
Nikolay Gospodinov	Federal Reserve Bank of Atlanta, USA
Uwe Hassler	Goethe-Universität Frankfurt, Germany
Javier Hidalgo	London School of Economics, UK
Kohtaro Hitomi	Kyoto Institute of Technology, Japan
Mehdi Hosseinkouchack	EBS Universität, Germany
Hsein Kew	Monash University, Australia
Jiwoong Kim	University of South Florida, USA
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Yun-Yeong Kim	Dankook University, South Korea
Hira L. Koul	Michigan State University, USA
Heejun Lee	Brown University, USA
Jungyoon Lee	University of London, Royal Holloway, UK
Han-Ying Liang	Tongji University, China
Yingqian Lin	Shanghai University of Finance and Economics, China
Alex Maynard	University of Guelph, Canada
Keiji Nagai	Yokohama National University, Japan
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Elena Pesavento	Emory University, USA
Peter C. B. Phillips	Yale University, USA; University of Auckland, New Zealand; Singapore Management University, Singapore; and University of Southampton, UK
Myung Hwan Seo	Seoul National University, South Korea
Yu Shen	Tongji University, China
Yixiao Sun	University of California, San Diego, USA
Masaya Takano	McGill University, Canada

Junfan Tao	Kyoto University, Japan
Yundong Tu	Peking University, China
Qiying Wang	The University of Sydney, Australia
Xiaohu Wang	Fudan University, China
Weilin Xiao	Zhejiang University, China
Jun Yu	Singapore Management University, Singapore
Ying Zhou	Monash University, Australia

INTRODUCTION

Volume 45 of *Advances in Econometrics* honors Professor Joon Y. Park, who has made numerous and substantive contributions to the field of econometrics over a career spanning four decades since the 1980s and counting. Volume 45 consists of 28 chapters and is in fact split between two volumes with the first focusing on econometric theory and the second focusing on econometric applications. These papers have been contributed by Joon's friends, colleagues, coauthors, former students, and even his dissertation advisor, Professor Peter C. B. Phillips, and the volume is edited by his wife and most frequent collaborator, Professor Yoosoon Chang, and two of his former students.

In the typical fashion of *Advances in Econometrics*, the papers were to be submitted in early 2021 after a conference in Joon's honor in April 2020, which would have nearly coincided with his 65th birthday. Of course, the COVID-19 pandemic forced much of the world into lockdown in April 2020, so plans changed. Papers were still submitted in 2021, but the conference was delayed and, as of this writing, is scheduled for September 29–30, 2023, in Bloomington, Indiana, which Joon and Yoosoon have called home for nearly 15 years.

We introduce the 13 chapters of the first volume, which are loosely grouped into three sections that are closely related to Professor Park's contribution to the theoretical analysis of time series and particularly related to the research of the first two or so decades of his career.

After graduating from Yale under the supervision of Professor Phillips, Joon's early work in the late 1980s and 1990s focused on nonstationary time series and particularly on cointegration and common stochastic trends, where some of his most highly cited contributions were made. These include foundational work on regressions with nonstationary series (Park & Phillips, 1988, 1989); the variable addition test for cointegration (Park, 1990), which remains one of the most highly cited papers in *Advances in Econometrics*; and perhaps his most well-known contribution to the field on canonical cointegrating regressions (CCR)[1] (Park, 1992).

Shifting his research, Park published a series of papers in the late 1990s and 2000s on nonlinear transformations of unit root processes, which introduce nontrivial obstacles in the form of nonstandard rates of convergence and limiting distributions. One could say that his work helped to redefine *nonstandard* in the sense that up to this point, nonstandard typically meant rate-T with limiting Dickey–Fuller type distributions. The rates of convergence in these papers generally involve powers of the sample size other than ½ or 1, and the limits usually include nonlinear functions of stochastic integrals and/or Brownian local times. Park's most well-cited contributions to the study of nonlinear transformation of nonstationary series are Park and Phillips (1999, 2001), but his work on nonlinearity has also spilled over into time-varying coefficients (Park & Hahn, 1999), instrumental variables (Chang et al., 2004), functional coefficients (Cai et al., 2009), and other areas.

Following the themes of nonstationarity and nonlinearity, the papers in this volume are grouped as follows: (I) nonstationarity, unit roots, and fractional noise; (II) nonlinearity; and (III) inference and prediction using models with trending series.

Part I: Nonstationarity, Unit Roots, and Fractional Noise

A contribution by Peter C. B. Phillips, not only Joon's dissertation advisor but also longtime editor of *Econometric Theory*, appropriately opens the volume on econometric theory and the section on nonstationarity, unit roots, and fractional noise. Specifically, his article "Discrete Fourier Transforms of Fractional Processes With Econometric Applications" presents an exact representation of the discrete Fourier transform in terms of the component data, which he finds to be particularly useful for analyzing the asymptotic behavior of the periodogram when the memory parameter exceeds the threshold for stationarity. He shows that smoothed periodogram spectral estimates remain consistent for frequencies away from the origin as long as the memory parameter is strictly less than unity.

Also studying fractional noise, Xiaohu Wang, Weilin Xiao, and Jun Yu contribute the article "Asymptotic Properties of the Least Squares Estimator in Local to Unity Processes With Fractional Gaussian Noises." They derive the asymptotic properties of the autoregressive parameter in local to unity processes with errors generated as fractional Gaussian noise with the Hurst parameter over the interval (0,1). The rates of convergence are standard rate-T over the upper half of this interval, but nonstandard and dependent of the Hurst parameter over the lower half. They derive limiting distributions over this interval that are new to the literature except at 1/2.

Critical to ascertaining stationarity or lack thereof are unit root tests. In their contribution, "Powerful Self-normalizing Tests for Stationarity Against the Alternative of a Unit Root," Uwe Hassler and Mehdi Hosseinkouchack introduce a new and powerful tool to address this well-known problem. Specifically, they propose a family of tests for stationarity against a local unit root that builds on the Karhunen–Loève expansions of the limiting CUSUM process under the null hypothesis and a local alternative. They find that the proposed tests are more powerful than the classic KPSS test.

Also on the topic of testing for unit roots, Kohtaro Hitomi, Keiji Nagai, Yoshihiko Nishiyama, and Junfan Tao contribute "A Sequential Test for a Unit Root in Monitoring a p-th Order Autoregressive Process." They study unit root tests for autoregressive processes of order p under sequential sampling schemes using stopping times based on the observed Fisher information. They derive the joint limit of the test statistics and the stopping time under the null and local alternatives, which are nonstandard.

Part II: Nonlinearity

As we mentioned, both cointegration and functional coefficients are areas in which Professor Park has made contributions to the literature. Han-Ying Liang, Yu Shen, and Qiying Wang contribute to the volume and this literature with "Functional-coefficient Cointegrating Regression With Endogeneity." As the title suggests, they explore nonparametric estimation of cointegrating regression models with functional coefficients and where the structural equation errors are serially dependent and the regressor is endogenous. In this context, they show the self-normalized local kernel and local linear estimators to be asymptotically normal.

In "A Specification Test Based on Convolution-type Distribution Function Estimates for Non-linear Autoregressive Processes," Kun Ho Kim, Hira L. Koul, and Jiwoong Kim develop a test for a parametric specification of the autoregressive function of a given stationary autoregressive time series. Their test is based on the integrated square difference between the empirical distribution function estimate and a convolution-type distribution function estimate of the stationary distribution function obtained from the autoregressive residuals.

Yingqian Lin and Yundong Tu contribute "Transformation Models With Cointegrated and Deterministically Trending Regressors," which contains important and interesting extensions of the statistical foundation for the nonlinear cointegrated models pioneered by Park and his coauthors. For a general transformation model with a time trend, stationary regressors, and unit root regressors, they estimate the transformation parameter and other model parameters by minimizing the concentrated loss function, and they obtain the asymptotic distributions of the proposed estimators.

The threshold model has been frequently used to model the nonlinearity of time series. Park and Shintani (2016) examine testing issues surrounding threshold effects and unit roots. In "Minimax Risk in Estimating Kink Threshold and Testing Continuity," Javier Hidalgo, Heejun Lee, Jungyoon Lee, and Myung Hwan Seo derive a risk lower bound in estimating the threshold parameter without knowing whether the threshold regression model is continuous or not. They show that the bound goes to zero as the sample size grows only at the cube root rate. Motivated by this finding, they develop a continuity test for the threshold regression model and a bootstrap to compute its p-values.

Part III: Inference and Prediction Using Models With Trending Series

Articles in the final section of this volume deal with models containing stochastic and/or deterministic trends, as do many of Professor Park's papers, from his earliest work on cointegration (Park & Phillips, 1988, 1989) and his widely cited CCR paper (Park, 1992) through his more recent work, such as that on estimating stochastic trends in state-space models (Chang et al., 2009).

In the first of these, "Semiparametric Independence Tests Between Two Infinite-order Cointegrated Series," Chafik Bouhaddioui, Jean-Marie Dufour, and Masaya Takano propose a semiparametric approach for testing independence between two cointegrated vector autoregressive series of infinite order. The residual-based tests allow for computational simplicity and weak assumptions on the form of the underlying process. The authors derive the asymptotic distributions of the test statistics under the null hypothesis and establish consistency of the tests against fixed alternatives of serial cross-correlation of unknown form.

Nikolay Gospodinov, Alex Maynard, and Elena Pesavento contribute "Inference in Conditional Vector Error Correction Models With a Small Signal-to-Noise Ratio," in which they study vector error correction models when the error correction term is characterized simultaneously by high persistence (near-unit-root behavior) and very small (near zero) variance. The importance of these features lies in the fact that conventional cointegration tests may fail to detect cointegration. The authors develop asymptotic theory for the parameter estimators for unconditional and conditional vector error correction models with these features.

Yixiao Sun, in his contribution entitled "Some Extensions of Asymptotic F and t Theory in Nonstationary Regressions," extends the asymptotic theory for F- and t-tests to linear regression models where the regressors could contain deterministic trends, unit-root processes, and near-unit-root processes. The tests themselves are implemented in the usual ways, but approximations to the limiting distributions are more accurate than the more commonly used chi-squared and normal approximations.

The last two contributions focus on predictive models with nonstationary series. Ying Zhou, Hsein Kew, and Jiti Gao contribute "Non-stationary Parametric Single-index Predictive Models: Simulation and Empirical Studies." Their model is designed to handle a wide variety of nonlinear relationships between the regressand and a single-index component containing either the cointegrated predictors or the non-cointegrated predictors. They introduce a new estimation procedure and investigate its finite-sample properties.

We opened the volume with a contribution from Joon's advisor, so it seems appropriate to close the volume with a contribution from one of his many students. In "Best Linear Prediction in Cointegrated Systems," Yun-Yeong Kim introduces the best linear predictor with the asymptotic minimum mean squared forecasting error among linear predictors of variables in cointegrated systems with unknown error specification. He suggests a switching predictor that automatically selects the random walk or cointegration model according to the size of the estimated autocorrelation coefficient estimated from the residuals.

We hope you enjoy reading "Essays in Honor of Joon Y. Park: Econometric Theory" and learning about the advances in econometrics made by the authors as much as we have!

NOTE

1. In case you have ever wondered… yes, Joon has always been a fan of the music of Credence Clearwater Revival, also abbreviated as CCR!

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PART I

NONSTATIONARITY, UNIT ROOTS, AND FRACTIONAL NOISE

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CHAPTER 1

DISCRETE FOURIER TRANSFORMS OF FRACTIONAL PROCESSES WITH ECONOMETRIC APPLICATIONS*

Peter C. B. Phillips[a,b,c,d]

[a] *Yale University, New Haven, CT, USA*
[b] *University of Auckland, Auckland, New Zealand*
[c] *Singapore Management University, Singapore*
[d] *University of Southampton, Southampton, UK*

ABSTRACT

The discrete Fourier transform (dft) of a fractional process is studied. An exact representation of the dft is given in terms of the component data, leading to the frequency domain form of the model for a fractional process.

* This paper has a long history. It was presented at the Cowles Foundation Conference "New Developments in Time Series Econometrics," October 23–24, 1999, and to the New Zealand Econometric Study Group, July 1999, Auckland, New Zealand. The paper originated in some notes on fractional processes in the nonstationary case that were written in May 1998 and circulated at Yale. It was completed in 1999 while the author was living on Waiheke Island and visiting the University of Auckland. That version (Phillips, 1999a) was conditionally accepted subject to revision by the *Annals of Statistics* but the revise-by-date expired and the revision was never submitted. The paper was revised and updated in 2021, leaving its main results essentially unchanged. Thanks go to the Editor and referees of the *Annals of Statistics*, Chang Sik Kim, Katsumi Shimotsu, Yixiao Sun, and Karim Abadir for comments on the original paper. Katsumi and Yixiao made further helpful comments on the revision, which are greatly appreciated. Thanks are due to the NSF for research support under Grant Nos. SBR 94-22922, 97-30295, and SES 18-50860.

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This representation is particularly useful in analyzing the asymptotic behavior of the dft and periodogram in the nonstationary case when the memory parameter $d \geq \frac{1}{2}$. *Various asymptotic approximations are established including some new hypergeometric function representations that are of independent interest. It is shown that smoothed periodogram spectral estimates remain consistent for frequencies away from the origin in the nonstationary case provided the memory parameter* $d < 1$. *When* $d = 1$, *the spectral estimates are inconsistent and converge weakly to random variates. Applications of the theory to log periodogram regression and local Whittle estimation of the memory parameter are discussed and some modified versions of these procedures are suggested for nonstationary cases.*

Keywords: Discrete Fourier transform; fractional Brownian motion; fractional integration; log periodogram regression; nonstationarity; operator decomposition; semiparametric estimation; Whittle likelihood;

JEL classification: C22

1. INTRODUCTION

Studies of nonstationary time series over the last four decades have produced a vast body of knowledge that has transformed the conduct of empirical research in economics. The impact of this research is now manifest in empirical work throughout the social and business sciences. A catalyst supporting these developments was the widespread recognition that real world processes in society, economics, and politics are influenced in fundamental ways by advances in technology, firm investments, and individual human decision-making. These processes are rarely, if ever, stationary. Inevitably they evolve in uncertain ways over time, reflecting the arrival of new shocks to the system, some of which have persistent effects. Recognizing this reality led to an understanding that methods of data analysis need to account for the fact that the way in which memory is carried in the data differs in a fundamental manner among stationary, near-stationary, and nonstationary processes.

Acknowledgment of the importance of this distinction is evident in early researches of statisticians and economists at the turn of the twentieth century (Hooker, 1901; Pearson & Elderton, 1923; Yule, 1926) on nonsense correlations[1] and the work of the mathematician Bachelier (1900) on speculative prices, which introduced the notion of a stochastic process. Methods began to emerge later that provided probabilistic underpinnings and foundations for statistical inference with data that demonstrated long range memory or dependence (Granger & Joyeux, 1980; Hosking, 1981; Hurst, 1951; 1956; Mandelbrot & Van Ness, 1968) and various types of random wandering behavior over time. In economics in the 1980s, advances in the use of function space limit theory were made that enabled the full trajectory features of nonstationary data to be reflected in regression asymptotics, leading to new understanding of such regressions, including both cointegrating and spurious regressions, and new methods of testing and inference for analyzing nonstationary data (Durlauf & Phillips, 1988; Phillips, 1986b; 1987; 1988; Phillips & Durlauf, 1986).

Joon Park played a big part in these developments, starting with his doctoral dissertation research and early research at Yale (Park & Phillips, 1988; 1989) and a sustained series of subsequent works that have helped to push out the envelope of econometric methodology for linear, nonlinear, and continuous time methods of analysis with nonstationary data. Many of these works have been jointly conducted with the present author in a longstanding collaboration that has been as pleasurable and special an academic fellowship as much as it has enriched this field of research.

My contribution to this symposium of works honoring Joon Park relates to his research on nonstationary processes and focuses on some of the defining properties of long range-dependent time series. The present work has a history reaching back more than two decades and it is hoped that a good part of its value is retained amidst the considerable body of work that has emerged since the original version of the paper (Phillips, 1999a) was written. The first contribution of this chapter is to provide an exact representation of the dft of a fractional process, which enables asymptotic analysis of its behavior and various functionals such as the periodogram in the nonstationary case when the memory parameter $d \geq \frac{1}{2}$. The methods reveal that smoothed periodogram spectral estimates remain consistent for frequencies away from the origin in the nonstationary case provided the memory parameter $d < 1$. When $d = 1$, the spectral estimates are inconsistent and converge weakly to random variates. Some useful applications of this theory are given for log periodogram regression and local Whittle estimation of the memory parameter in nonstationary cases. For an advanced textbook treatment of long memory processes, readers are referred to Surgailis et al. (2012).

The plan of this chapter is as follows. Various preliminaries are given in the following Section 2. Some useful new decompositions and representations in the frequency domain are developed in Section 3 that extend related decompositions in the time domain. Section 4 develops asymptotic approximations for dfts involving special functions that help to simplify representations and enable development of limit theory for dfts of fractional processes in nonstationary cases. These results extend earlier work on the limit theory of dfts of stationary processes to the fractional case. For higher levels of dependence, when $d = 1$, the leakage from the zero frequency becomes dominant and affects the limit theory at all frequencies, so that dfts are spatially correlated across frequency asymptotically, quite unlike the stationary case. Section 5 provides some applications of the results to spectral estimation and to semiparametric estimation of the memory parameter. Particular attention in the latter case is given to log periodogram regression and local Whittle estimation. Some modified versions of these procedures are suggested which conveniently extend their range of applicability to the nonstationary case. Final remarks on long memory and autoregressive approaches to nonstationarity close out Section 5. Proofs and technical results are in the Appendix in Section 6. A notational summary is given at the end of this chapter in Section 7.

A final word of introduction. While our focus is on the case where $d \in \left(\frac{1}{2},1\right)$, the methods introduced here are applicable when $d>1$, and in modified form when $|d|<\frac{1}{2}$. A particularly useful approach is to combine the exact representation (3.7) that applies when $d=1$ with results for fractional d to produce valid representations for the $d>1$ case. The remarks and results in paragraphs 3.6–3.8 indicate some of these possibilities.

2. PRELIMINARIES

We consider the fractional process X_t generated by the model

$$(1-L)^d X_t = u_t, \quad t=0,1,... \tag{2.1}$$

Our interest is primarily in the case where X_t is nonstationary and $d \geq \frac{1}{2}$, so in (2.1) we work from a given initial date $t=0$, set $u_j=0$ for all $j \leq 0$, and assume that $u_t(t>0)$ is stationary with 0 mean and continuous spectrum $f_u(\lambda)>0$. This formulation corresponds to a Type II fractional process (Davidson & Hashimzade, 2009; Marinucci & Robinson, 1999). Expanding the binomial in (2.1) gives the form

$$\sum_{k=0}^{t} \frac{(-d)_k}{k!} X_{t-k} = u_t, \tag{2.2}$$

where

$$(d)_k = \frac{\Gamma(d+k)}{\Gamma(d)} = (d)(d+1)\ldots(d+k-1)$$

is Pochhammer's symbol for the forward factorial function and $\Gamma(\cdot)$ is the gamma function. When d is a positive integer, the series in (2.2) terminates, giving the usual formulae for the model (2.1) in terms of differences and higher order differences of X_t. An alternate form for X_t is obtained by inversion of (2.1), giving

$$X_t = (1-L)^{-d} u_t = \sum_{k=0}^{t} \frac{(d)_k}{k!} u_{t-k}. \tag{2.3}$$

Throughout this chapter it will be convenient to assume that the stationary component u_t in (2.1) is a linear process of the form

$$u_t = C(L)\varepsilon_t = \sum_{j=0}^{\infty} c_j \varepsilon_{t-j}, \quad \sum_{j=0}^{\infty} j\left|c_j\right| < \infty, \quad C(1) \neq 0, \tag{2.4}$$

for all t and with $\varepsilon_t = iid\left(0,\sigma^2\right)$ with finite fourth moments. The summability condition in (2.4) is satisfied by a wide class of parametric and nonparametric models for u_t, enables the use of the techniques in Phillips and Solo (1992), and ensures that partial sums of u_t satisfy a functional central limit theorem, which will be needed later.

Under (2.4), the spectrum is $f_u(\lambda) = \frac{\sigma^2}{2\pi}\left|\sum_{j=0}^{\infty} c_j e^{ij\lambda}\right|^2$ and $f_u(0) = \frac{\sigma^2}{2\pi}C(1)^2 > 0$.[2] In view of (2.1), it is natural to define

$$f_x(\lambda) = |1 - e^{i\lambda}|^{-2d} f_u(\lambda). \tag{2.5}$$

The function $f_x(\lambda)$ gives the spectrum of X_t when it exists and X_t is stationary (i.e., for $|d| < \frac{1}{2}$ and under infinite past initialization of X_t in (2.3)) and is the analog of the spectrum in the nonstationary case when $d \geq \frac{1}{2}$ even though it is not integrable. In that case, Solo (1992) gave a formal justification of $f_x(\lambda)$ as a spectrum in terms of the limit of the expectation of the periodogram. Taking logarithms of (2.5) produces the equation

$$\ln(f_x(\lambda)) = -2d\ln(|1 - e^{i\lambda}|) + \ln(f_u(\lambda)), \tag{2.6}$$

which motivates a linear log periodogram regression for the estimation of d, in which $f_x(\lambda)$ is replaced by periodogram ordinates $I_x(\lambda)$ evaluated at the fundamental frequencies $\lambda_s = \frac{2\pi s}{n}$, $s = 0,1,\ldots,n-1$. Here, $I_a(\lambda_s) = w_a(\lambda_s)w_a(\lambda_s)^*$, $w_a(\lambda_s)$ is the dft, $w_a(\lambda_s) = \frac{1}{\sqrt{2\pi n}}\sum_{t=1}^{n} a_t e^{it\lambda_s}$ of a time series a_t, and w^* is the complex conjugate of w. With this substitution (2.6) becomes

$$\ln\left(I_x(\lambda_s)\right) = -2d\ln\left|1 - e^{i\lambda_s}\right| + \ln\left(f_u(\lambda_s)\right) + U(\lambda_s), \tag{2.7}$$

where $U(\lambda_s) = \ln\left[I_x(\lambda_s)/f_x(\lambda_s)\right]$. By virtue of the continuity of f_u, $f_u(\lambda_s)$ is effectively constant for frequencies in a shrinking band around the origin, suggesting a linear least squares regression of $\ln\left(I_x(\lambda_s)\right)$ on $\ln\left|1 - e^{i\lambda_s}\right|$ over frequencies $s = 1,\ldots,m$ (with m a truncation number). The method has undoubted appeal, is easy to perform in practice and has been commonly employed in applications. However, (2.6) is a moment condition, not a data generating mechanism, and the analysis of this regression estimator is complicated by the difficulty of characterizing the asymptotic behavior of the dft $w_x(\lambda_s)$, which is the central element in determining the properties of the regression residual $U(\lambda_s)$ in (2.7).

An important contribution by Künsch (1986) showed that, for fractional processes like (2.1), $w_x(\lambda_s)$ has quite different statistical properties from the corresponding dft, $w_u(\lambda_s)$, of the stationary process u_t for frequencies in the immediate neighborhood of the origin. In particular, for $\lambda_s = \frac{2\pi s}{n} \to 0$, with s fixed as $n \to \infty$ the dft ordinates are asymptotically correlated, not uncorrelated. Analyses by Robinson (1995b) and Hurvich et al. (1998) for Gaussian u_t have provided an asymptotic theory in the stationary case, thereby placing log periodogram regression on a rigorous footing. More recent work has dealt with nonstationary cases where $d \geq \frac{1}{2}$ (Kim & Phillips, 2006; Phillips, 2007; Velasco, 1999). Another semiparametric estimation procedure, suggested by Künsch (1987), is the Gaussian estimator which maximizes a local version of the Whittle likelihood, which is known to have a smaller variance than log periodogram regression in the stationary case (Robinson, 1995a). This estimator also relies on the behavior of $w_x(\lambda_s)$ for frequencies in the vicinity of the origin. More recent work on Whittle estimation has focused on nonstationary cases where $d \geq \frac{1}{2}$ (Abadir et al., 2007; Phillips, 2014; Phillips & Shimotsu, 2004; Shao, 2010; Shimotsu & Phillips, 2005; 2006; Velasco & Robinson, 2000) and cases of noise contaminated data (Sun & Phillips, 2003) such as in the estimation of the Fisher equation (Sun & Phillips, 2004).

This chapter provides new methods for studying the asymptotic behavior of $w_x(\lambda_s)$ for nonstationary values of d. The approach relies on an exact representation of $w_x(\lambda_s)$ in terms of the dft $w_u(\lambda_s)$ and certain residual components. This representation aids in the analysis of the properties of $w_x(\lambda_s)$ and, thereby, in the study of log periodogram regression and local Whittle estimation. The representation also provides a frequency domain version of the data generating mechanism (2.1) above. As such, it is useful in motivating some alternative approaches to inference about d that are proposed here and which have been explored in subsequent work that has appeared since the first version of this chapter circulated in 1999.

3. FREQUENCY DOMAIN DECOMPOSITIONS

It is convenient to manipulate the operator $(1-L)^d$ in (2.1), with its polynomial expansion (2.2), in a form that more readily accommodates dfts. This can be done algebraically, as in Phillips and Solo (1992), by expanding the polynomial operator about its value at the complex exponential $e^{i\lambda}$, leading to the following decomposition.

3.1. Lemma

Define the fractional operator expansion $D_n(L;d) = \sum_{k=0}^{n} \frac{(-d)_k}{k!} L^k$. *Then*

$$D_n(L;d) = D_n\left(e^{i\lambda};d\right) + \tilde{D}_{n\lambda}\left(e^{-i\lambda}L;d\right)\left(e^{-i\lambda}L-1\right), \tag{3.1}$$

where $\tilde{D}_{n\lambda}\left(e^{-i\lambda}L;d\right) = \sum_{p=0}^{n-1} \tilde{d}_{\lambda p} e^{-ip\lambda} L^p$ *and* $\tilde{d}_{\lambda p} = \sum_{k=p+1}^{n} \frac{(-d)_k}{k!} e^{ik\lambda}$.

The representation (3.1) is an immediate consequence of formula (32) in Phillips and Solo (1992) and can be obtained by straightforward algebraic manipulation. No summability conditions are required here for its validity since it is a finite sum. However, the value of d does affect the order of the terms in this expansion and, consequently, the order of magnitude of these terms when $n \to \infty$,