From Counting to Computing

Ideas for Mathematics Education in Information Age

Sergei Abramovich

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By

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CONTENTS

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ABOUT THE AUTHOR

Sergei Abramovich (Ph.D., Mathematics) has been an instructor to more than 4,000 prospective K-12 teachers of mathematics over the span of more than 30 years. He is the author/co-author/editor of 13 books published by Information Age Publishing (2), Nova Science Publishers (4), Springer (3), and World Scientific (4), and some 250 journal articles, book chapters, and conference proceedings on mathematics education and mathematics. His service to educational community includes the founding editor-in-chief appointment with Advances in Educational Research and Evaluation, Singapore, and membership on editorial boards of another seven professional journals in Australia, China (Hong Kong), England, Russia, and the United States.

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PREFACE

The book reflects the author's experience using educational technology in the preparation of K-12 teachers of mathematics for American and Canadian schools. The university where the author works for more than 25 years is in Upstate New York on the border with Canada, thereby attracting many Canadians pursuing their master's degree in education. The intent of the book is to demonstrate and promote the appropriate use of technology (both physical and digital), something that integrates formal mathematical reasoning, tactile educational experiment, and digital computation through solving problems. The focus of problem-solving activities of the book is on numeric tables shaped as squares, equilateral and isosceles triangles, and square-shaped staircases. Such tables provide ample opportunities for algebraic generalization in information age. Numeric background for the activities involves addition and multiplication tables, Pascal's triangle, the natural number sequence and its various subsequences, including polygonal numbers. The book is based on the idea that simple counting of objects arranged in geometric shapes not only leads to the development of numeric patterns but can be extended to digital computing when those objects are filled with integers defined by various numeric sequences and the shapes that the objects form determine computational algorithms. A didactical framework (called technology-immune/technology-enabled) of integrating mathematical argument, pattern (both visual and symbolic) recognition, and digital computation in problem solving is used throughout the book. This framework serves as a conceptual shortcut toward achieving a variety of mathematically consequential computational outcomes, allowing one to avoid the complexity of, otherwise required, mathematical machinery. The book covers several classic topics from arithmetic, number theory, combinatorics, and probability theory. Many historical connections, cultural origins, and teaching methods of those topics are highlighted including the names of Pythagoras, Aristotle, Heron of Alexandria, Theon, Fibonacci, Gersonides, Pacioli, Cardano, Galilei, Kepler, Descartes, Fermat, Pascal, Spinoza, Leibniz, Jacob Bernoulli, Pestalozzi, Binet, de Moivre, Lamé, Lucas, Dewey, Vygotsky, Wertheimer, and Pólya. The book can be used in mathematics teacher education programs, both elementary and secondary, as well as in courses on discrete and algorithmic mathematics demonstrating how simple algorithms and commonly available digital tools such as an electronic spreadsheet, *Wolfram Alpha* (https://www.wolframalpha.com/; the "Pro for Educators" version was used throughout the book), *Maple* (Char et al., 1991), and the *Graphing Calculator* (Avitzur, 2011), can achieve sophisticated and conceptually notable computational outcomes.

The book consists of eight chapters. In the first chapter titled *From Concepts to Conceptual Shortcuts to the Use of Technology* basic mathematical concepts and activities such as one-to-one correspondence, counting, base-ten system, comparison of integers are mentioned. The notion of conceptual shortcut as an effective way of mathematical reasoning is introduced. The use of technology is presented as a conceptual shortcut allowing one to focus on big ideas, instead of dealing with technicalities of symbolic transformations, when solving mathematical problems. Examples of using *Wolfram Alpha* and *Maple* as tools of conceptual shortcuts in finding integer-sided triangles with given perimeter and deriving Binet's formula for Fibonacci numbers are presented. Technology-immune/technology-enabled (TITE) problem-solving framework to be used throughout the book is introduced. A tactile solution to a problem with a broken key on a calculator is discussed and the use of the visual versus the computational is interpreted as a conceptual shortcut in solving the problem. A historical geometric problem with more than one correct answer from the time of Pythagoras is presented.

The second chapter titled *Exploring the Addition and the Multiplication Tables* begins with a brief history of numeric tables and their potential for motivating conceptual explorations. The search for perfect squares in the addition and the multiplication tables is presented through the lens of the TITE framework. Using a spreadsheet-based random number generator, the large size tables are explored demonstrating that the number of perfect squares in the addition table is greater than in the corresponding multiplication table. The addition table and spreadsheet modeling are used for comparing theoretical and experimental probabilities of rolling two dice. The classic notion of gnomon is introduced to include a shape, a number, and an algebraic expression. Connection of the sum of numbers in a gnomon of an addition table to pentagonal numbers is discussed. Square size tables (grids) are considered as an arrangement of unit squares with unitary fillings (i.e., empty squares) to enable their geometric representations. Partitioning of a large square into a combination of a smaller square and three gnomons inside the former square is considered both from visual and computational perspectives. Addition and multiplication tables are considered as the arrangement of unit squares filled, respectively, with addition and multiplication facts. Identities established through counting in the context of unitary fillings are modified through non-unitary fillings, thereby opening a window to computing. Various computations demonstrate the ubiquity of the number six in the context of the tables. A TITE problem with historical flavor supported by physical (virtual) manipulatives, *Wolfram Alpha*, and a spreadsheet is presented.

The third chapter titled *Exploring Equilateral Triangles Filled with Integers* considers an equilateral triangle built out of counters as a new type of table in which a row of counters is a gnomon of such a table. The TITE framework is used to support activities with such tables. Summation activities, like those discussed in the second chapter, have been carried out and computationally supported by *Maple* and *Wolfram Alpha*. Equilateral triangles, the shapes of which determine computational algorithms, built out of counters filled with consecutive triangular, square, pentagonal, and hexagonal numbers are considered. Identities among the sums of numbers appearing in different parts of the triangles have been developed using the TITE framework supported by *Wolfram Alpha* and *Maple*.

In the fourth chapter titled *Exploring Isosceles Triangles Filled with Integers* isosceles triangles built out of counters each row of which has two or three more counters than the previous row and filled with consecutive natural, odd, triangular, and square numbers are considered as a new type of a numeric table. Using the TITE framework, summation activities within the triangles, like those discussed in the context of equilateral triangles, have been carried out and computationally supported by *Maple* and *Wolfram Alpha*. The shape of an isosceles triangle determines the sequences of numbers assigned to its lateral sides. Those sequences govern computational algorithms of finding different sums within the triangle. Isosceles triangles built out of counters filled with consecutive odd and triangular numbers in which there is one counter in the first row (the vertex), and each other row has m more counters than the previous row is considered. Relations among the sums of different parts of the triangle have been explored. The activities are computationally supported by *Wolfram Alpha* and *Maple*.

In the fifth chapter titled *Exploring Squares Filled with Integers* evolving staircases built out of counters forming consecutive $n \times n$ and $(2n - 1) \times (2n - 1)$ squares starting from $n = 1$ (to allow for the first step be a single counter) and filled with consecutive triangular numbers are considered. Identities involving the last and the first triangular numbers in two neighboring squares are formulated and proved using the TITE framework. Activities are supported by *Wolfram Alpha* and *Maple*. Using the TITE framework, the summation formula for consecutive triangular numbers assigned to counters forming an $n \times n$ square is developed and proved using the TITE framework. Evolving staircases forming consecutive $n \times n$ squares, $n \geq 1$, the counters of which filled with consecutive square and pentagonal numbers are considered. An identity involving the last and the first square numbers in two consecutive squares is proved using the TITE framework. Although it is shown how activities can be supported by *Maple* and *Wolfram Alpha*, in the case of squares a point is made that habitual reliance on technology without a critical technology-immune (TI) perspective can lead to an unnecessary technology-enabled (TE) standpoint. A summation formula for square numbers assigned to counters forming the $n \times n$ square is developed. The formula is proved using the TITE framework. *Wolfram Alpha* is used to generate the sum. Two identities involving the last and the first square numbers and pentagonal numbers, respectively, included in counters forming an odd size consecutive squares have been developed. The identities are proved using the TITE framework. Computations are supported by *Wolfram Alpha* and *Maple*.

The sixth chapter titled *Pascal's Triangle as a Bridge from Combinatorics to Probability* begins with highlighting combinatorial problem solving as a new direction in mathematical pedagogy of the second part of the 20th century allowing students to extract the concepts of mathematics from real-life situations. The importance of formal reasoning as part of the TITE framework used in this chapter is highlighted. A method of reduction to a simpler problem is considered in the context of putting identical finger rings on fingers. The Rule of Sum as a theoretical basis of tactile activities enables the construction of a spreadsheet for the finger rings problem which is considered as a combinatorial problem. The Rule of Product and the permutation of letters in a word are discussed using concrete materials. It is shown how the Rule of Sum can be used as an alternative to the Rule of Product and a few basic combinatorial identities have been developed. The appearance of and connections among triangular, tetrahedral, and pentatope numbers is mentioned in the context of the spreadsheet constructed. The case when all finger rings are different is considered and connected to Pascal's triangle. The case of two fingers and distinct finger rings is described in terms of the binomial expansion. The sum of coefficients in the expansion of the expression $(a_1 + a_2 + ... + a_k)^n$ is considered for different values of k (fingers) and n (finger rings). Pascal's triangle is interpreted as a tool used by Pascal in recording the incidents of coin tossing. It is shown how Bernoulli formula, allowing one to find the probability of k successes in n trials with exactly two possible outcomes, can be derived from Pascal's triangle. The formula is applied to randomly putting different finger rings on two fingers. The chapter concludes with a classic probability problem of the Division of Stakes and a spreadsheet for posing such problems is constructed.

The seventh chapter titled *From Pascal's Triangle to Fibonacci-Like Polynomials* begins with connecting Pascal's triangle to Fibonacci numbers and interpreting the powers of two as gnomons either through subtraction or through addition. A special structural modification of Pascal's triangle leads to exploring the difference between the nth power of two and the Fibonacci number of rank $n + 1$. The entries of the rows of the modified Pascal's triangle are used to construct polynomials with special properties, called Fibonacci-like polynomials. The concept of generalized Golden Ratios as strings of numbers of different lengths is connected to Fibonacci-like polynomials the coefficients of which form rows in the structurally modified Pascal's triangle. The properties of the polynomials and their roots are illustrated using computer graphing. A two-parametric linear difference equation of the second order is considered and reduced to a non-linear recursive equation the solutions of which may form cycles. Continued fractions are used to find the relations among the parameters that produce cycles. Geometrically, such relations are parabolas sharing the vertex at the origin. It is shown that plugging the coordinates of any point on a parabola into a spreadsheet which models the non-linear equation numerically confirms the existence of a cycle formed by the solutions of the equation with those values of parameters. Fibonacci-like polynomials are described as gnomons either through addition or through subtraction. A Fibonacci-like polynomial of degree three has been developed by solving a continued fraction equation using *Maple*. It is shown that two irrational roots of this polynomial provide proper cycles of length eight, whereas the integer root provides a cycle of length four (as a trivial cycle of length eight). The chapter ends with an open question about what happens with solutions of the two-parametric difference equation outside parabolas defined by the roots of Fibonacci-like polynomials.

In the final eighth chapter titled *Problems Motivated by Digital Computing* two types of problems on divisibility of polynomial expressions depending on an integer variable and appearing in other chapters of the book are considered. In both cases, the TITE framework is followed by using its TE part for computational demonstration of divisibility. The problems of the first type allow for the demonstration of divisibility by transforming a dividend into a clear multiple of a divisor using such basic facts as the product of two consecutive integers is a multiple of two and the product of three consecutive integers is a multiple of six. The use of the basic facts represents a TI part of proof. This approach does not require demonstration that transition from n to $n + 1$ upholds the inductive assumption made regarding the case of n. The second type of problems seeking proof of divisibility involves the method of mathematical induction based on the transition from n to $n + 1$, something that requires computational support by *Wolfram Alpha*. Some proofs may require using the *Maple*-based mathematical induction proof. One can see that the second type of problems does have a strong TE component. In some cases, a TI part may precede a TE part by transforming a polynomial to a form that can be computationally treated in a more effective way. In other words, a TI part serves as a conceptual shortcut toward more effective use of technology as a TE part.

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CHAPTER 1

FROM CONCEPTS TO CONCEPTUAL SHORTCUTS TO THE USE OF TECHNOLOGY

ABSTRACT

This chapter reviews basic mathematical concepts and activities such as one-to-one correspondence, counting, base-ten system, and comparison of integers. The notion of conceptual shortcut as an effective way of mathematical reasoning is introduced. The use of technology is presented as a conceptual shortcut allowing one to focus on big ideas instead of dealing with technicalities of symbolic transformations when solving mathematical problems. Examples of using *Wolfram Alpha* and *Maple* as tools of conceptual shortcuts in finding integer-sided triangles with given perimeter and deriving Binet's formula for Fibonacci numbers are presented. Technology-immune/technology-enabled problem-solving framework to be used throughout the book is introduced. A tactile solution to a problem with a broken key on a calculator is discussed and the use of the visual vs. the computational is interpreted as a conceptual shortcut in solving the problem. A historical geometric problem with more than one correct answer from the time of Pythagoras is presented.

Keywords: Counting; one-to-one correspondence; base-ten system; conceptual shortcut; big ideas; technology

From Counting to Computing, pages 1–30

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1.1 Introduction

What is a mathematical concept? Why do we need to teach mathematics conceptually? Assuming that a concept is preceded by an idea, the latter often stems from observing concrete instances happening around us and recognizing their similarities and differences. One of the basic concepts in mathematics is the concept of number which can be introduced through the activity of counting. For example, two baskets of apples and oranges may have same number of fruits and, if so, we can recognize similarity between the baskets; when there are more apples than oranges, we can recognize difference between the baskets. The latter outcome may not necessarily result from direct counting. With this in mind, consider a group of children and a basket with apples from which each child should be given an apple. The idea is to do it physically through what may be called one-to-one correspondence: this is a child, and this is an apple for the child. Through this process, one can understand that depending on a situation, the concreteness of one-to-one correspondence can be replaced by counting children and apples to see whether the quantities of objects—children and apples—make it possible to decide if each child can be given an apple. This idea brings about the concept of natural (counting) number—a common characteristic of different sets of objects that can be used in application to the counting of objects and to comparing their quantities.

Counting, however, requires certain skills not included in a pretty intuitive action of making one-to-one correspondence between two sets of objects. Among those skills is the proficiency of knowing number names (labels assigned to objects), their order, and that the last label (a number name) used in counting signifies the total number of objects counted (starting the counting with the name one). The next step in developing the concept of number deals with comparison. How do we compare numbers? How do we know that the number 6 is greater than the number 5? There are several ways to decide which number is bigger. When the two numbers are abstracted from their origin, knowing the order of names used in counting can be a way to say that 6 is greater than 5. Making one-to-one correspondence between apples (6) and children (5) is another way to decide—because one apple is left through this process, 6 is greater than 5. One can use identical (in size) square tiles to represent the quantities 6 and 5 to see that one tower (train) of tiles is taller (longer) than the other tower (train). At the same time, five apples appear being bigger than six blueberries thus justifying the use of identical square tiles as tools of comparison.

But one can easily imagine large quantities of apples and children when the above two strategies, counting and one-to-one correspondence, are not effective in deciding which quantity is larger; for example, how do we know that 654 is greater than 456? Assuming that we know such large number names, counting up to the latter number and recognizing that the former number was not used in counting is not a tenable method of comparison. This may be one of the first examples demonstrating the need for mathematics and the value of its concepts and notation in real life. Using fingers to count may prompt the idea of base-ten system (see below Figure 1.16) to be used in counting: 10 fingers is a new counting unit, 10 such units is another counting unit, and so on. In that way, the number 456 points at 6 fingers, 5 groups of 10 fingers, and 4 groups of the unit which consists of 10 groups with 10 fingers in each. This perspective on the meaning of the symbolism of natural numbers would help one to compare 456 with 654 in a tenable way, the result of which can even be seen as conceptual discovery. Indeed, as Gottfried Wilhelm Leibniz (1646–1716, the great German mathematician and philosopher) remarked, “It is worth noting that the notation facilitates discovery” (cited in Zeidler, 2011, p. 439).

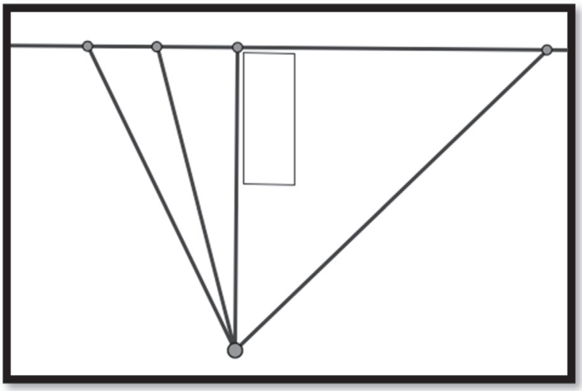
One may inquire: is there the largest natural number (assuming that one knows the smallest number)? To answer this question by creating a frame of reference, one has to imagine that if such number exists, it describes a certain set of objects which apparently includes all objects that exist—rocks, books, animals, people, and so on. But the number of objects can be changed by adding the very set of all objects so constructed to this set the cardinality of which is described by the “largest” number and therefore increasing such number by one. One can say that this would be the largest number that exists as there is nothing left to be added. This position is difficult to refute thereby indicating that what we call infinity does not have a clear frame of reference and it represents a truly abstract concept. That is, in order to imagine that a number can always be increased (or, likewise, decreased) requires one to be capable of abstract thinking.

1.2 Conceptual Shortcuts and Geometric Intuition

Common Core State Standards (2010) expect mathematics students of all ages to be “able to compare the effectiveness of two plausible arguments” (p. 7) when solving problems. Plausible arguments originate within a classroom environment in which students are not afraid to say something that they see reasonable even when others might not see it that way. It is through comparison of different arguments that insight, prompting a conceptual shortcut, can emerge. Supporting students’ natural curiosity can trigger such insight. One can identify the use of conceptual shortcut in solving problems with insight (Dunker, 1945), or productive thinking (Wertheimer, 1959), or displacement of concepts (Schon, 1963). A conceptual shortcut aims at providing problem solving with an effective way of reasoning leading to an answer. Often, experience in problem solving

Figure 1.1

Measuring Distance From Point to Line



may lead to using an effective strategy. For example, how can one measure distance from a point on the floor to the wall? Although it is intuitively clear that perpendicular provides the shortest distance from a point to the wall (Figure 1.1), young children can discover (or confirm) this experimentally by measuring distances along different segments and then deciding which one provides the shortest distance. As Johannes Kepler (1571–1630, a German astronomer and mathematician) would have said, "without proper experiments I conclude nothing" (cited in Burtt, 1925, p. 50). While the word perpendicular may be too difficult for young children to use, they can see that a piece of bank paper fits completely, with no gap or overlap, between the bottom border of the wall and the segment that provides the shortest distance. If someone asks about a name for such segment, then the name, perpendicular, can be provided.

1.2.1 Using Conceptual Shortcut in Arithmetical Calculations

The use of conceptual shortcut, the term used in a number of publications within the last three decades (Canobi, 2005; Eaves et al., 2019; Godau et al., 2014; Paliwal & Baroody, 2020; Robinson & LeFevre, 2012; Stern, 1992), demonstrates a more effective way of solving a problem based on conceptual understanding which, in turn, points at creativity in problem solving. These publications dealt mostly with the so-called three-term problems ($a + b - c$) in arithmetic offered to elementary school children to see if they can recognize that doing subtraction before doing addition as a conceptual shortcut, results in computationally easier solution.

To clarify, consider the case of using conceptual shortcut when computing a numeric expression 53 + 9 – 7. Figure 1.2 shows the use of manipulatives in solving this problem without conceptual shortcut. Here circles and triangles represent, respectively, ones and tens. If one follows the order in which operations are written, one has to add 9 to 53 that involves addition with regrouping; that is, to add 9 circles to 3 circles, then to create a group of 10 circles (out of 12 circles), trade 10 circles for 1 triangle and place this new triangle to the existing 5 to have 6 triangles and 2 circles. Now, one has to subtract 7 from 62; in other words, to subtract 7 circles from 2 circles (enhanced by 6 triangles one of which would allow for subtraction with borrowing). To do that, one has to trade one triangle for 10 circles to have, once again, 12 circles from which 7 circles must be subtracted to have 5 circles and 5 triangles. That is, 53 + 9 – 7 = 55. The use of manipulatives (something that reflects work of the mind) shows that this process is quite cumbersome.

At the same time, Figure 1.3 shows doing arithmetic through a conceptual shortcut by first subtracting seven circles from nine circles to have two circles which then are added to three circles on the far-left place value chart to have five circles and five triangles. Note that through the use of the conceptual shortcut the triangles were not touched; in other words, despite having a two-digit number, all arithmetic was carried out by the mind within units. In this example, the value of using a conceptual shortcut was obvious—"complex" computations were avoided by noting that more simple computations can be carried out. The simplicity of those computations can be compared to the use of technology in assisting a problem solver in doing symbolic computations that often are truly complex. In order to use technology as a conceptual shortcut, one must understand what kind of technology

Figure 1.2

53 + 9 – 7 = 62 – 7 = 55

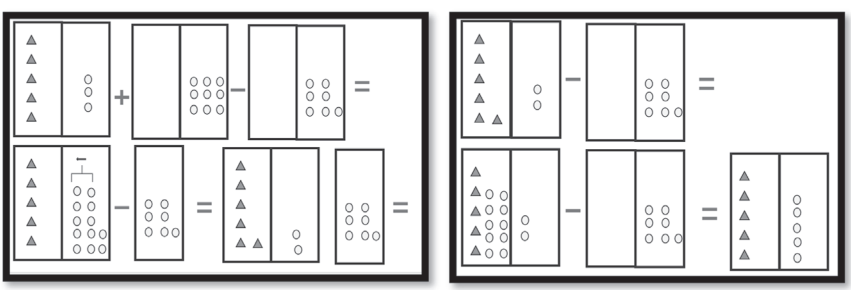
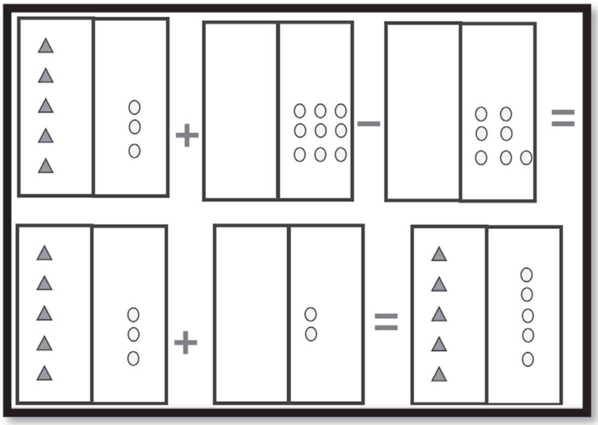


Figure 1.3

53 + 9 − 7 = 53 + 2 = 55



to use, what kind of programming code to use, how to make this code computationally efficient, and so on. This preliminary step in using technology as a conceptual shortcut requires the use of formal reasoning that is not based on the use of technology; in other words, it is immune of technology. This brings about the concept of technology-immune/technology-enabled (TITE) problem-solving framework (see Section 1.4).

Remark 1.1. Note that in the case of 55 + 7 – 9 subtracting 9 from 7 is not possible until grade 5 at the earliest when negative numbers allow for the subtraction of a larger number from a smaller number. Alternatively, one has to know that $7 - 9 = -(9 - 7) = -2$. So, in the $a + b - c$ problem conceptual shortcut is either not possible or its use depends on the grade level.

1.2.2 Using Conceptual Shortcut Beyond Arithmetic

Many uses of conceptual shortcuts can be identified beyond arithmetic (Abramovich, 2022; Kuo et al., 2013). The very notion of learning mathematics through the use of conceptual shortcuts can be applied to more advanced topics and the use of computing technology can be seen as a shortcut serving to evade complicated procedures of traditional mathematics. As was mentioned at the outset of the *Teaching Machines* movement in the United States, "mechanical aids are possible which … would leave the teacher more free for her most important work, for developing in her pupils fine enthusiasms, clear thinking and high ideas" (Pressey, 1926,

p. 376). Also, it will be demonstrated that non-digital (tactile) tools can be used to support conceptual shortcuts in mathematical problem solving.

Henriksen et al. (2018) indicated that effectiveness in solving a problem is a key feature of creativity. At the same time, a formal proof of the accuracy of a conceptual shortcut may be mathematically complex, yet its real-life application is pretty obvious. Teaching young children to use conceptual shortcuts can help them develop "the growth mind set" (Conference Board of the Mathematical Sciences, 2012, p. 9), through which it is easier to "make sense of problems and persevere in solving them" (Common Core State Standards, 2010, p. 6).

Consider another geometric situation that can be seen through the lens of conceptual shortcut. It deals with the concept known as the triangle inequality—*the sum of any two side lengths of a triangle is greater than the third side length.* Often, people apply this inequality (without being consciously aware of it) when crossing across the grass rather than waking along the pavement to get to a building faster. In other words, one applies a conceptual shortcut within a real-life situation. It is only by interpreting this shortcut in terms of the inequality, i.e., by making an epistemic advancement in the learning of mathematics, that one becomes aware of this concept. This is similar to how one correctly uses grammar in conversation before learning formal rules of native language in grade school. In the words of Vygotsky (1987), "conscious awareness enters through the gate opened up by the scientific concept" (p. 191). At the same time, many mathematical concepts are not intuitive, and their study is similar to the study of a foreign language—a concept is studied first and applied later. For example, knowledge that among all rectangles with a given perimeter square has the largest area is not intuitive. Even less intuitive is knowledge that among all rectangles with a given area square has the smallest perimeter. Yet, application of those relationships between perimeter and area of rectangles is common in real life.

That is, a mathematical concept may stem from the need to support real-life activities. This points at the importance of concrete problems as pedagogical tools in the teaching of mathematics. Concreteness makes a student confident in the usefulness of the subject matter to be studied; it helps one "to recognize a mathematical concept in, or to extract it from a given concrete situation" (Pólya, 1965, pp. 100–101).

1.3 Use of Technology as a Conceptual Shortcut

Nowadays, the appropriate use of digital technology (that includes conceptual understanding of the intricacies of computational power it offers) can be seen as a shortcut enabling one to circumvent the complexity of paper and pencil symbolic computations. For example, finding the number of integer-sided rectangles with given perimeter $2p$ (by solving a two-variable equation $x + y = p$, something that for small values of p can done through counting unit squares forming a rectangle) can be outsourced to a tool capable of symbolic computations like *Wolfram Alpha*, yet when asking the tool to find all integer-sided triangles with given perimeter (by solving a three-variable equation), one has to reduce the number of solutions provided by the tool using the triangle inequality (mentioned in Section 1.2.2). As shown in Figure 1.4, *Wolfram Alpha* provides eight positive integer solutions to the equation $a + b + c = 10$ under the condition $a \geq b \geq c$, out of which, as shown in Figure 1.5, only two may serve as side lengths of a triangle due to the added condition $a < b + c$ representing the triangle inequality among the side lengths of the triangle. One can see that already the triple (5, 3, 2) may not represent the side lengths of a triangle as $5 = 3 + 2$. This example suggests that, in general, when outsourcing problem solving to a digital tool, one has to possess some initial understanding of how the

Figure 1.4

Solving an Equation Without Attention to the Triangle Inequality