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OF ROAD PRICING**

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# MATHEMATICAL AND ECONOMIC THEORY OF ROAD PRICING

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# ACRONYMS

AC	Average Cost
BLPP	Bi-Level Programming Problem
BOT	Build-Operate-Transfer
BPR	Bureau of Public Roads
BLVI	Bi-Level Variational Inequality
CN	Cournot-Nash
DUE	Deterministic User Equilibrium
EB	Economic Benefit
EPEC	Equilibrium Problem with Equilibrium Constraints
KKT	Karush-Kuhn-Tucker
LP	Linear Program
MC	Marginal Cost
MPEC	Mathematical Program with Equilibrium Constraints
OD	Origin -Destination
SC	Social Cost
SO	System Optimum
SW	Social Welfare
STEN	Space-Time Expanded Network
SUE	Stochastic User Equilibrium
UB	User Benefit
UE	User Equilibrium
VI	Variational Inequality
VOT	Value Of Time



# GLOSSARY OF NOTATION

$a$	a link $a \in A$
$A$	the set of all links
$\bar{A}$	the subset of links subject to a toll charge
$\tilde{A}$	the subset of links with observed flows
$B_w(d_w)$	the benefit function between OD pair $w \in W$
$B_w^m(d_w^m)$	the benefit function of user class $m \in M$ between OD pair $w \in W$
$c_{rw}$	the travel cost on path $r$ connecting OD pair $w \in W$
<b>c</b>	the vector of path travel costs
$c_{rw}^m$	the travel cost on path $r$ between OD pair $w$ by users of class $m \in M$
$C_a$	the capacity of link $a \in A$
$d_w$	the travel demand between OD pair $w \in W$
<b>d</b>	the vector of OD demands
$\hat{D}_w$	the potential demand or the upper-bound of demand between OD pair $w \in W$
$d_w^m$	the travel demand of user class $m$ between OD pair $w \in W$
$\hat{D}_w^m$	the potential demand of user class $m$ between OD pair $w \in W$
$D_w(\mu_w)$	the demand function between OD pair $w \in W$
$D_w^{-1}(d_w)$	the inversed demand function between OD pair $w \in W$
$E_v^u$	Elasticity of $v$ to $u$ , $E_v^u = (u/v) dv(u)/du$
$f_{rw}$	the flow on path $r \in R_w$
<b>f</b>	the vector of path flows
$f_{rw}^m$	the flow of user class $m \in M$ on path $r \in R_w$ between OD pair $w \in W$
$G$	$(N, A)$ be a network with node set $N$ and link set $A$
$K$	the set of Cournot-Nash (CN) players or the set of vehicle types
$I_a(y_a)$	the construction cost of link $a$ as a function of link capacity $y_a$
$m$	a user class $m \in M$
$M$	the set of all user classes with different values of time
$N$	the set of all nodes
$r$	a path or a route $r \in R$
$R_w$	the set of all paths connecting OD pair $w \in W$

$R$	$= \bigcup_{w \in W} R_w$
$t_a^0$	the free-flow travel time (moving time on link $a \in A$ )
$t_a(v_a)$	the travel time on link $a \in A$ as a function of link flow $v_a$
$\tilde{t}_a(v_a)$	the marginal social travel time on link $a \in A$ , including the congestion externality, $\tilde{t}_a(v_a) = t_a(v_a) + v_a dt_a(v_a)/dv_a$
$t_a(v_a, y_a)$	the link travel time as a function of both flow and capacity variables
$\mathbf{t}(\mathbf{v})$	the vector of link travel times
$u_a$	toll charge on link $a \in A$
$\mathbf{u}$	the vector of link tolls
$v_a$	the flow on link $a \in A$
$\mathbf{v}$	the vector of link flows
$v_a^m$	the flow of user class $m$ on link $a \in A$
$v_a^k$	the link flow of vehicle type $k \in K$ or CN player $k \in K$
$w$	an OD pair $w \in W$
$W$	the set of all OD pairs
$W^k$	the set of OD pairs of which users are controlled by CN player $k \in K$
$W^K$	$= \bigcup_{k \in K} W^k$
$W^m$	the set of OD pairs of which users are controlled by UE class $m \in M$
$W^M$	$= \bigcup_{m \in M} W^m$
$y_a$	the capacity of a new link $a$ or capacity increase of an existing link $a$
$\mathbf{y}$	the vector of link capacity variables
$\beta$	the users' value of time
$\beta_m$	the average value of time of user class $m \in M$
$\delta_{ar}$	1, if link $a \in A$ is on path $r \in R$ and 0, otherwise
$\Delta$	the link/path incidence matrix, $\Delta = [\delta_{ar}]$
$\varepsilon$	a predetermined tolerance for stopping iterative process
$\Lambda_{rw}$	1 if path $r \in R_w$ and 0, otherwise
$\Lambda$	the OD/path incidence matrix, $\Lambda = [\Lambda_{rw}]$
$\mu_w$	the Lagrange multiplier associated with the demand conservation constraint of OD pair $w \in W$ , or the generalized travel cost between OD pair $w \in W$ at equilibrium
$\mu_w^m$	the generalized travel cost of user class $m \in M$ between OD pair $w \in W$
$\mu$	the vector of generalized OD travel costs

$\Omega$	the feasible region of link flows and OD flows
$\Omega_f$	the feasible region of path flows
$\Omega_v$	the feasible region of link flows
$\Omega^k$	the feasible region of links flows of CN player $k \in K$
$\Omega^m$	the feasible region of link flows of a UE class $m \in M$
$\Omega^U$	the feasible region of link flows of UE player
$\nabla_v \mathbf{t}(\mathbf{v})$	the Jacobian matrix of link travel time function $\mathbf{t}$ with respect to $\mathbf{v}$
$\in$	element membership
$\subset$	proper set inclusion
$\cup$	union of sets
$\infty$	infinity
$ \cdot $	the cardinality of a finite set
$\ \cdot\ $	the Euclidean norm
$\arg \min_x F(x)$	the set of $x$ attaining the minimum of the function $F(x)$



# PREFACE

Road pricing as an effective means of both managing road traffic demand and raising additional revenue for road construction has been studied extensively by both transportation researchers and economists. Practical implementation has been progressing rapidly and electronic road pricing schemes have been proposed and tested worldwide. It is likely that over the next few years, with increasing public acceptability, there will be greater use of road pricing. The incontestable fact is that there is a great need for the development of efficient road-use pricing models.

Despite a few monographs and journal special issues in the literature that have been devoted to the topic in recent years, most studies have focused on empirical studies, policy experience, environmental issue of road congestion and road pricing. There is still scope for methodological development of advanced road pricing systems, such as dynamic pricing, integrated road and transit pricing, as well as practical toll charging schemes in general road networks. This book is intended to deal with a number of timely topics including: fundamentals of user-equilibrium problems; principle of marginal-cost pricing applied to road pricing; existence of optimal link tolls for system optimum under multi-class, multi-criteria, multiple equilibrium behaviors; social and spatial equity issues in road pricing; optimal design of private toll roads; simultaneous determination of optimal toll levels and locations; trial-and-error implementation of marginal-cost pricing on networks with unknown demand functions; dynamic road pricing.

This book would appear to be the first book devoted exclusively to the mathematical and economic investigation of road-use pricing in general congested networks, which aims at alleviating traffic congestion, improving transport conditions and enhancing social welfare. It constitutes an update of the state of the art of the latest research, mainly by the authors and their colleagues. The book is targeted at students, professionals and scientists who are studying and working in relevant transportation fields.

We are most thankful to a number of individuals for their help during our work in this field. First and foremost we would like to acknowledge the contribution and continuing encouragement of Prof. M.G.H. Bell of Imperial College London (ICL). In fact, the manuscript was finalized during the first author's sabbatical leave at ICL in early 2005, and he is particularly grateful to Prof. Bell for his hospitality during a very fruitful and enjoyable stay in ICL. Thanks are also due to Prof. R. Lindsey of the University of Alberta, Prof. E.

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# 1

## INTRODUCTION

### 1.1 BACKGROUND

Roadway congestion is a source of enormous economic costs. In principle, many of these costs can be prevented, as they result from socially inefficient choices by individual drivers. A number of regions have considered alleviating roadway congestion by introducing congestion pricing. Indeed, road pricing has become one of the priorities on transport policy agendas throughout the world. It is increasingly believed that road pricing may offer an effective instrument to manage travel demand, and to raise revenue that may, for instance, be used for transport improvements. An increasing number of congestion pricing schemes have been proposed, tested or implemented worldwide. Examples include the US's value pricing scheme, recent EU green and white papers, Dutch initiatives, electronic road pricing schemes in Singapore and Hong Kong, and the London congestion charging scheme that was introduced in February 2003.

It is conceivable that a new generation of road-use pricing technologies will be widely considered for introduction on many congested road networks. All these emerging technologies offer efficient tools and challenges for implementation of road pricing for traffic management, and also generate a great need for the development of efficient pricing models that allow for the evaluation and perhaps design of innovative road pricing schemes. Some monographs and journal special issues have been devoted to these topics in recent years. However, the majority of studies have focused on the empirical aspects, policy experiences, and environmental issues of road congestion and road pricing. There is, however, still ample scope for methodological developments for the analysis of advanced road pricing systems, such as dynamic pricing, integrated road and transit pricing, as well as practical toll charging schemes in general road networks.

### 1.2 THEORETICAL DEVELOPMENTS

The initial idea of road pricing was put forward by Pigou (1920), who used the example of a congested road to make points on externalities and optimal congestion charges (see, also, Knight, 1924, for early interpretation of social cost and toll charge). The fundamental concept

is easy: to apply a price mechanism in the same way as it applies elsewhere in a market economy. Prices should be higher under congestion conditions and lower at less congested times and locations in order to deter excessive uses. The key question is how to choose the appropriate price in a simple yet practical manner, under what is often a complex set of economic and technical circumstances. The key issue from a practical point of view is how to implement the concept, not only in terms of developing technically efficient charging mechanisms, but also in gaining political acceptance as a valid policy instrument (Button and Verhoef, 1998).

There have been both intellectual and practical developments after Pigou's idea of using road pricing measures to regulate road traffic congestion. Seminal works on road pricing that need to be mentioned include Wardrop (1952), Walters (1961), Beckmann (1965) and Vickrey (1969). The subject of road pricing has particularly rejuvenated interest extensively from both economists and transportation researchers in recent years, because of the growing prominence and changing nature of urban transportation problems facing a modern city. Here we do not intend to fully review the large body of literature (which will be done in individual chapters) but merely outline the various types of road pricing problems of interest and some relevant analytical studies. Readers are invited to refer to McDoland (1995), Lo and Hickman (1997), Button and Verhoef (1998), Verhoef (1996), McDonald et al. (1999), Levinson (2002), Yang and Verhoef (2004), and Santos (2004) for excellent reviews or monographs on the subject, from the economic perspective.

### **1.2.1 The First-best Pricing Problem**

The theoretical background of road-use pricing has relied upon the fundamental economic principle of marginal-cost pricing, which states that road users using congested roads should pay a toll equal to the difference between the marginal-social cost and the marginal-private cost in order to maximize the social surplus. By so doing, each user will face the marginal social cost of road use other than the marginal private costs. For a congested road, this includes the value of time losses imposed on other road users, as well as the value of emissions, noise and accident risks created. The social surplus, defined as the difference between the total benefits and total costs, is often considered as an appropriate indicator for social welfare, and its maximization is a condition for economic efficiency to prevail (Verhoef, 1996).

The theory of the marginal-cost pricing or the first-best pricing in the literature by Pigou and followers (for example, Walters, 1961; Vickrey, 1963; Evans, 1992; Hills, 1993) was developed based on the demand-supply (or performance) curves for the standard case of a homogenous traffic stream moving along a given uniform stretch of road, such as an expressway, connecting given entry and exit points. There would be some decision on what

the optimal traffic flow should be, essentially determined by estimating the traffic speed-flow relationship underlying the performance curve. The congestion charge would then be an efficient way of attaining the target flow of traffic.

In the case of homogeneous users, the first-best congestion pricing theory is established in general traffic networks. In line with this theory, a toll that is equal to the difference between the marginal social cost and the marginal private cost is charged on each link, so as to internalize the user externalities and thus achieve a system optimum flow pattern in the network (Beckmann, 1965; Dafermos and Sparrow, 1969, 1971). Investigations have been conducted on how this classical economic principle would work in a general congested road network with multiple vehicle types, such as trucks versus passenger cars (Dafermos, 1972, 1973), with both multiple vehicle types and link flow interactions (Smith, 1979b), with queuing (Yang and Huang, 1998), and in a congested network in a stochastic equilibrium (Yang, 1999). Moreover, Bellei et al. (2002) developed a variational inequality model for network pricing optimization in a multi-user and multi-modal context; Liu and Boyce (2002) presented a variational inequality formulation of the marginal-cost pricing problem for a general transportation network with multiple time periods. Recently, Yang, Zhang and Meng (2004) developed a trial-and-error implementation method for marginal-cost pricing on networks in the absence of demand functions.

### 1.2.2 The Second-best pricing problems

In spite of its perfect theoretical basis, the first-best or marginal-cost pricing scheme is of little practical interest. The problem stems from the fact that it is impractical to charge users on each network link in view of the operating cost and public acceptance. Note that a simple introduction of the “marginal-cost pricing tolls” over a *subset* of the links of a network based on a local link speed-flow relation may distort the allocation of the traffic over the entire network and may cause degradation instead of improvement in social welfare. Thus, setting the toll equal to the marginal cost of the trips may not be valid for the design of an optimal road pricing scheme in a more complex environment.

Due to the imperfection of the first-best pricing, the second-best pricing is, in fact, often the most relevant case from a practical perspective and has received ample attention in the recent literature (Lindsey and Verhoef, 2001). The key questions to be answered in a second-best pricing scheme in general include: where to levy the toll and how much? What different impacts would the pricing schemes bring to different users?

Most studies in the context of the second-best pricing problems are generally conducted with respect to the determination of toll levels for given charge locations. The simplest version of the problem concerns the two route problem, where an untolled alternative road is available